

# SUPERATOMIC BOOLEAN ALGEBRAS: MAXIMAL RIGIDITY

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ABSTRACT. We prove that for any superatomic Boolean Algebra of cardinality  $> \beth_\omega$  there is an automorphism moving uncountably many atoms. Similarly for larger cardinals any of those results are essentially best possible

## ANOTATED CONTENT

### §1 Superatomic Boolean Algebra have nontrivial automorphism

[We prove that if  $B$  is a superatomic Boolean Algebra, then it has quite a nontrivial automorphism; specifically if  $B$  is of cardinality  $> \beth_4(\sigma)$  then  $B$  has an automorphism moving  $> \sigma$  atoms. We then discuss how much we can weaken the superatomicity assumptions.]

### §2 Constructing counterexamples

[Under some assumptions we construct examples of superatomic Boolean Algebras for which every automorphism move few atoms.]

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*Key words and phrases.* Set Theory, Boolean Algebras, superatomic, rigid; pcf, MAD.

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§3 Sufficient conditions for the existence of  $\langle B_i : i < \mu \rangle$

[We deal with the assumptions of the construction in §2.]

## §0 INTRODUCTION

We show that for a superatomic Boolean Algebra has an automorphism moving uncountably many atoms if it is large enough, really  $> \beth_u$ ; similarly replacing  $\aleph_0$  by  $\theta$ ; (an automorphism move an atom if its image is not itself). We then show that those results are essentially best possible. Of course, we can express those results in topological terms. See [M] and his cite on background and history, in particular work of Rubin and work of the author.

**Notation**

**0.1 Definition.** 1) For a Boolean Algebra  $B$  let us define the ideal  $\text{id}_\alpha(B)$  by induction:

$$\text{id}_0(B) = \{0\}$$

$$\text{id}_\beta(B) = \{x_1 \cup \dots \cup x_n : \text{for some } \alpha < \beta \text{ and } n < \omega \text{ for each } \ell \in \{1, \dots, n\} \text{ the element } x_\ell / \text{id}_\alpha(B) \text{ is an atom of } B / \text{id}_\alpha(B) \text{ or } x_\ell \in \text{id}_\alpha(B)\}.$$

Hence for limit  $\delta$  we have

$$\text{id}_\delta(B) = \bigcup_{\beta < \delta} \text{id}_\beta(B) \text{ for limit } \delta.$$

$$\text{Let } \text{id}_\infty(B) = \bigcup_{\alpha} \text{id}_\alpha(B).$$

- 2) For  $x \in \text{id}_\infty(B)$  let  $\text{rk}(x, B) = \text{Min}\{\alpha : x \in \text{id}_{\alpha+1}(B)\}$ .  
 3)  $B$  is superatomic if  $B = \text{id}_\infty(B)$  and  $\text{dp}(B)$  be the ordinal  $\alpha$  such that  $B / \text{id}_\alpha(B)$  is a finite Boolean Algebra.

## §1 SUPERATOMIC BOOLEAN ALGEBRA'S HAVE NONTRIVIAL AUTOMORPHISMS

**1.1 Theorem.** *Assume*

- (a)  $B$  is a superatomic Boolean Algebra with no automorphism moving  $\geq \theta$  atoms; that is if  $\pi$  is an automorphism of  $B$  then  $\{x : x \in \text{atom}(B) \text{ and } \pi(x) \neq x\}$  is a set of cardinality  $< \theta$
- (b)  $\theta$  regular uncountable.

Then  $|B| \leq \beth_3(2^{<\theta})$ , so if  $\theta = \sigma^+$  then  $|B| \leq \beth_4(\sigma)$ .

*Proof.* Toward contradiction let  $B$  be a counterexample and let  $\mu$  be the number of atoms of  $B$ . Without loss of generality

- $\boxtimes_1$   $B$  is a Boolean Algebra of subsets of  $\mu$  and its atoms are the singletons  $\{\alpha\}, \alpha < \mu$ .

Let  $I =: [\mu]^{<\theta} \cap B = \{x \in B : |x| < \theta\}$ , clearly  $I$  is an ideal of  $B$  and let  $Y =: \{x : x \in B \text{ and } x/I \text{ is an atom of } B/I\}$ .

We shall prove (after some preliminary things) that:

- $\boxtimes_2$  if  $x \in Y$  then  $|x| \leq 2^{(2^{<\theta})}$ .

We shall say that a set  $a \subseteq \mu$  is  $B$ -autonomous if  $(\forall y \in I)(y \cap a \in B)$ ; in this case we let  $B \upharpoonright a = B \cap \mathcal{P}(a)$ .

Clearly

- $\otimes_1$  the family of  $B$ -autonomous subsets of  $\mu$  is a Boolean ring of subsets of  $\mu$  (i.e. closed under  $a \cap b, a \cup b, a \setminus b$ ) and include  $I$
- $\otimes_2$  for a  $B$ -autonomous  $a$ ,  $B \upharpoonright a$  is a Boolean algebra of subsets of  $a$  which include  $\{\{\alpha\} : \alpha \in a\}$ .

Also

- $\otimes_3$  if  $a_0, a_1$  are  $B$ -autonomous subsets of  $\mu$ ,  $x \in Y$ ,  $a_0 \subseteq x$ ,  $a_1 \subseteq x$  and  $B \upharpoonright a_0 \cong B \upharpoonright a_1$  over  $B \upharpoonright (a_1 \cap a_2) = B \cap \mathcal{P}(a_1 \cap a_2)$ , then there is an automorphism  $h$  of  $B$  such that  $h$  maps  $a_0$  to  $a_1$ ,  $a_1$  to  $a_0$  and  $\alpha \in \mu \setminus a_0 \setminus a_1 \Rightarrow h(\{\alpha\}) = \{\alpha\}$ .

[Why? Let  $g$  be an isomorphism from  $B \upharpoonright a_0$  onto  $B \upharpoonright a_1$  over  $B \upharpoonright (a_0 \cap a_1)$ ; now we define a permutation  $h$  of  $\text{atom}(B) = \{\{\alpha\} : \alpha < \mu\}$ ; let  $\alpha \in a_0 \Rightarrow h(\{\alpha\}) = g(\{\alpha\})$ ,  $h(g(\{\alpha\})) = \{\alpha\}$  and  $\alpha \in \mu \setminus a_0 \setminus a_1 \Rightarrow h(\{\alpha\}) = \{\alpha\}$ , by the demands on  $g$  clearly  $h$  is a well defined permutation of  $\text{atom}(B)$ . Now

$h$  can be naturally extended to an automorphism  $\hat{h}$  of  $\mathcal{P}(\mu)$  of order two, we have to check that  $\hat{h}$  maps  $B$  onto itself; even into itself suffice (because of “order two”). Clearly  $\hat{h}(x) = x$  and  $\hat{h} \upharpoonright (B \upharpoonright (\mu \setminus x))$  is the identity. So it is enough to check : that  $\hat{h} \upharpoonright (B \upharpoonright x)$  is an automorphism of  $B \upharpoonright x$ . But  $I \cap (B \upharpoonright x)$  is a maximal ideal of  $B \upharpoonright x$  (as  $x \in Y$ ) hence it is enough to check that  $\hat{h}$  maps  $I \cap (B \upharpoonright x)$  into itself. As  $b \in I \cap (B \upharpoonright x) \Rightarrow b = (b \setminus a_0 \setminus a_1) \cup (b \cap a_0 \cap a_1) \cup (b \cap a_0 \setminus a_1) \cup (b_1 \cap a_1 \setminus a_0)$ , and all four are in  $I$ ; clearly it is enough to check the following statements:  $b \in I$  &  $b \subseteq x \setminus a_0 \setminus a_1 \Rightarrow h(b) \in I$ , and  $\ell < 2$  &  $b \in I$  &  $b \subseteq x \cap a_\ell \setminus a_{1-\ell} \Rightarrow \hat{h}(b) \in I$  and lastly  $b \in I$  &  $b \subseteq a_0 \cap a_1 \Rightarrow \hat{h}(b) \in I$ . The second implication holds by the choice of  $g$ , the first as  $\hat{h}(b) = b$  in this case and the last one as  $h \upharpoonright (a_0 \cap a_1)$  is the identity so again  $\hat{h}(b) = b$ .]

- ⊗<sub>4</sub> if  $b \subseteq \mu, |b| \leq 2^{<\theta}$  then for some  $B$ -autonomous set  $c$  we have  $a \subseteq c \subseteq \mu, |c| \leq 2^{<\theta}$ .  
 [Why? Find  $c$  satisfying  $b \subseteq c \subseteq \mu, |c| \leq 2^{<\theta}$  such that  $(\forall y \in [c]^{<\theta})[(\exists z)(y \subseteq z \in I) \rightarrow (\exists z \subseteq c)(y \subseteq z \in I)]$ , just close  $\theta$  times recalling  $\theta$  is regular. Now if  $g \in I$  then  $|g| < \theta$  hence  $g \cap c \in [c]^{<\theta}$  so there is  $z$  such that  $g \cap c \in z \in I$  &  $z \subseteq c$ ; hence  $y \cap c = y \cap z \in I$ . This proves that  $c$  is autonomous. Now check.]

Now we return to the promised  $\boxtimes_2$ .

*Proof of  $\boxtimes_2$ .* if  $|x| > 2^{2^{<\theta}}$  let  $\alpha_i \in x$  for  $i < (2^{(2^{<\theta})})^+$  be pairwise distinct, let  $a_i$  be  $B$ -autonomous set of cardinality  $\leq 2^{<\theta}$  such that  $\{\alpha_{i+\varepsilon} : \varepsilon < 2^{<\theta}\} \subseteq a_i$  (exists by  $\otimes_4$ ), and without loss of generality  $a_i \subseteq x$  (just use  $a_i \cap x$ ). For some club  $C$  of  $(2^{2^{<\theta}})^+$ , we have  $i < j \in C \Rightarrow a_i \cap \{\alpha_{j+\varepsilon} : \varepsilon < 2^{<\theta}\} = \emptyset$  hence  $i < j \in C \Rightarrow |a_j \setminus a_i| \geq 2^{<\theta}$ . Now  $I \cap \mathcal{P}(a_i)$  has cardinality  $\leq |a_i|^{<\theta} \leq 2^{<\theta}$  (as  $\theta$  is regular) but  $x \in Y$  hence  $B \upharpoonright a_i$  has cardinality  $\leq 2^{<\theta}$ , hence the number of isomorphism types of  $B \upharpoonright a_i$  is  $\leq 2^{(2^{<\theta})}$ . Hence there is a stationary  $S \subseteq \{\delta < (2^{(2^{<\theta})})^+ : \text{cf}(\delta) = (2^{<\theta})^+\}$  and  $a^*$  such that  $i \in S$  &  $j \in S$  &  $i \neq j \Rightarrow a_i \cap a_j = a^*$  (the  $\Delta$ -system lemma). Also the number of isomorphic types of  $(B \upharpoonright a_i, \alpha)_{\alpha \in a^*}$  is at most  $\leq 2^{(2^{<\theta})}$  hence for some  $i < j$  from  $C \cap S$  we have  $B \upharpoonright a_i \cong B \upharpoonright a_j$ , but  $|a_j \setminus a_i| \geq 2^{<\theta} \geq \theta$  hence by  $\otimes_3$  there is an automorphism  $h$  of  $B$  which moves  $\geq 2^{<\theta}$  atoms, contradiction.  
 Next

- $\boxtimes_3$   $|Y/I| \leq \beth_2(2^{<\theta})$ .  
 [Why? If not, we can find  $x_i \in Y$  for  $i < (\beth_2(2^{<\theta}))^+$  such that  $i \neq j \Rightarrow x_i/I \neq x_j/I$ . As  $|x_i| \leq \beth_1(2^{<\theta})$  by  $\boxtimes_2$ , by the  $\Delta$ -system lemma for some unbounded  $A \subseteq (\beth_2(2^{<\theta}))^+$  the set  $\langle x_i : i \in A \rangle$  is a  $\Delta$ -system hence without loss of generality  $\langle x_i : i \in A \rangle$  are pairwise disjoint (not really

needed just clearer). As  $B \restriction x_i$  is a Boolean Algebra of cardinality  $\leq \beth_1(2^{<\theta})$  (as  $I \cap \mathcal{P}(x_i)$  is a maximal ideal of  $B \restriction x_i$  and  $I \cap \mathcal{P}(x_i) \subseteq [x_i]^{<\theta}$  and  $|x_i| \leq \beth_1(2^{<\theta})$  by  $\boxtimes_2$ ) there are at most  $\beth_2(2^{<\theta})$  isomorphism types of  $B \restriction x_i$ . So for some  $i \neq j$  in  $A$  we have  $B \restriction x_i \cong B \restriction x_j$ , so as in the proof of  $\boxtimes_3$  there is an automorphism  $h$  of  $B$  mapping  $x_i$  to  $x_j$  hence moving  $\geq |x_i \setminus x_j| \geq \theta$  atoms because  $x_i \neq x_j \pmod I$ .]

Choose a set  $\{x_\alpha : \alpha < \alpha^* \leq \beth_2(2^{<\theta})\}$  of representatives of  $Y/I$  and let  $x^* = \bigcup_{\alpha < \alpha^*} x_\alpha$ , so  $x^* \subseteq \mu$ ,  $|x^*| \leq \beth_2(2^{<\theta})$ .

Define  $J = \{a \in B : a \cap x^* = \emptyset\}$ .

$\boxtimes_4$   $J \subseteq I$ .

[Why? If not, there is  $x \in J \setminus I$  such that  $x/I$  is an atom of  $B/I$  so  $x/I \in \{x_\alpha/I : \alpha < \alpha^*\}$ , so for some  $\alpha$ ,  $x/I = x_\alpha/I$  hence  $|x \setminus x_\alpha| < \theta$  hence  $|x \cap x_\alpha| = \theta$  hence  $x \cap x^* \neq \emptyset$  hence  $x \notin J$ , a contradiction.]

Define an equivalent relation  $\mathcal{E}$  on  $B$  :  $y_1 \mathcal{E} y_2$  iff  $y_1 \cap x^* = y_2 \cap x^*$ . Clearly  $\mathcal{E}$  has  $\leq 2^{|x^*|}$  equivalence classes and  $2^{|x^*|} \leq \beth_3(2^{<\theta})$ ; also  $y_1 \mathcal{E} y_2 \rightarrow y_1 \setminus y_2 \in J$  (see its definition). Choose a set of representatives  $\{y_\gamma : \gamma < \gamma^*\}$  for  $\mathcal{E}$  so  $\gamma^* \leq \beth_3(2^{<\theta})$  and let  $B^*$  be the subalgebra of  $B$  which  $\{y_\gamma : \gamma < \gamma^*\}$  generates. So  $|B^*| \leq \beth_3(2^{<\theta})$  and, being superatomic, the number of ultrafilters of  $B^*$  is also  $\leq \beth_3(2^{<\theta})$ . Next  $B$  is generated by  $J \cup B^*$  as for  $y \in B$  there is  $\gamma$  such that  $y \mathcal{E} y_\gamma$  and  $y_\gamma \in B^*$ ,  $y - y_\gamma \in J$ ,  $y_\gamma - y \in J$  hence  $y \in \langle J \cup B^* \rangle$ . For  $D$  an ultrafilter of  $B^*$  let  $Z_D = \{\alpha \in \mu : (\forall y \in B^*)(\alpha \in y \leftrightarrow y \in D)\}$ .

Clearly

$\boxtimes_5$  for every  $\alpha \in \mu \setminus x^*$  there is a unique ultrafilter  $D = D[\alpha]$  on  $B^*$  such that  $\alpha \in Z_D$  (and the number of such ultrafilters is  $\leq \beth_3(2^{<\theta})$ ).

Now

$\boxtimes_6$   $\mu \leq \beth_3(2^{<\theta})$ .

[Why? Assume that not. By  $\boxtimes_4$  for each  $i < \mu$  we can find a  $B$ -autonomous  $a_i$  such that  $|a_i| \leq 2^{<\theta}$  and  $[i, i + 2^{<\theta}) \subseteq a_i$ ; let  $a_i = \{\beta_{i,\varepsilon} : \varepsilon < \varepsilon_i\}$  with  $\beta_{i,\varepsilon}$  increasing with  $\varepsilon$ . Without loss of generality for some unbounded  $A \subseteq (\beth_3(2^{<\theta}))^+$  for all  $i \in A$  the following does not depend on  $i$  :  $\varepsilon_i$  and  $D[\beta_{i,\varepsilon}]$  for  $\varepsilon < \varepsilon_i$  (use  $\boxtimes_5$ ), and  $\{u \in [\varepsilon_i]^{<\theta} : \{\beta_{\varepsilon,i} : i \in u\} \in I\}$ , the  $\varepsilon$  such that  $\beta_{i,\varepsilon} = i$  and without loss of generality for  $j < i$  in  $A$ ,  $a_j \cap [i, i + 2^{<\theta}) = \emptyset$ . By the  $\Delta$ -system lemma without loss of generality for some  $a^*$  we have: for  $i < j$  in  $A$ ,  $a_i \cap a_j = a^*$ . So by  $\boxtimes_1$  the set  $a^*$  is  $B$ -autonomous and also  $a_i \setminus a^*$  is so we can use  $a_i \setminus a^*$ , so without loss of generality for  $i \neq j$  in  $A$ ,  $a_i \cap a_j = \emptyset$  and as  $|x^*| \leq \beth_3(2^{<\theta})$  clearly without loss of generality  $i \in A \Rightarrow a_i \cap x^* = \emptyset$ .

So for  $i \neq j$  in  $A$  there is an automorphism of  $B$  interchanging  $a_i, a_j$  (the proof is like that proof of  $\otimes_3$  using “ $B$  is generated by  $J \cup B^*$ ”). So we get a contradiction.]

So  $|J| \leq |[\mu]^{<\theta}| = \mu^{<\theta} \leq (\beth_3(2^{<\theta}))^{<\theta} = \beth_3(2^{<\theta})$  so as  $B$  is generated by  $J \cup B^*$  together we get the desired conclusion.  $\square_{1.1}$

*1.2 Discussion.* 1) We can weaken the assumption “ $B$  is superatomic by  $B/I_{<\theta}[B]$  is superatomic”, where: for a Boolean Algebra  $B$  and infinite cardinal  $\theta$  we define  $I_{<\theta}(B) = \{x \in B : B \restriction x \text{ has density } < \theta\}$  (see a little in [Sh 397, §1]). For  $B$  superatomic this is the  $I$  in the proof of 1.1 and the proof is the same. What if we just assume “ $B/I_{<\theta}[B]$  is atomic”? One point in the proof may fail: the number of ultrafilters of  $B^*$  is not  $\leq |\text{atom}(B^*)| \leq \beth_3(2^{<\theta})$  but is  $\leq 2^{|B^*|} \leq 2^{2^{|\mathcal{Y}|}} \leq \beth_4(2^{<\theta})$ , so we should replace  $\beth_3(2^{<\theta})$  by  $\beth_4(2^{<\theta})$  in the conclusion. We can adapt 2.1 to this case: e.g. let  $\langle d_\zeta : \zeta < \lambda = 2^\mu \rangle$  be a family of subsets of  $\mu$  such that any finite Boolean combination of them is infinite and let  $B^*$  be the Boolean subalgebra of  $\mathcal{P}[\mu]$  generated by  $\{c_\alpha : \alpha < \lambda = 2^\mu\} \cup \{\{i\} : i < \mu\}$ . We let  $\lambda' = 2^\lambda$  let  $\{c_\gamma : \gamma < \lambda'\}$  be an independent family of subsets of  $\lambda$  and we let  $X^* = \bigcup_{\alpha < \mu} X_\alpha \cup \{X_\gamma^* : \gamma < \lambda'\}$ ? We ignore  $A'$  (and omit clause (k) of the assumption) and among the generators of  $\mathbf{B}$ , clause (i), (ii) remains and

(iii)'  $c_\zeta = \{x \in X : \text{for some } \alpha \in d_\zeta \text{ we have } x \in X_\alpha\} \cup \{X_\gamma^* : \zeta \in C_\gamma, \gamma \in [\mu, \lambda']\}$ .

## §2 CONSTRUCTING COUNTEREXAMPLES

We would like to show that the bound from §1 is essentially best possible. The construction (in 2.1) is closely related to the proof in §1, but we need various assumptions. We shall deal with them later.

**2.1 Lemma.** *Assume*

- (a)  $\theta \leq \kappa \leq \mu \leq \lambda' \leq \lambda, \aleph_0 < \theta, \theta = \text{cf}(\theta) = \sigma \geq \aleph_1$
- (b) *there is an  $\mathcal{A} \subseteq [\mu]^{\aleph_0}$  almost disjoint (i.e.  $A \neq B \in \mathcal{A} \Rightarrow |A \cap B| < \aleph_0$ ) such that  $(\forall A \in [\mu]^\sigma)(\exists B \in \mathcal{A})(B \subseteq^* A)$  and  $|\mathcal{A}| = \mu$*
- (c)  $\bar{B} = \langle B_\alpha : \alpha < \mu \rangle$
- (d)  $B_\alpha$  is a superatomic Boolean Algebra with  $\leq \kappa$  atoms such that any automorphism of  $B_\alpha$  moves  $< \theta$  atoms and  $|B_\alpha| \leq \lambda$ ; moreover if  $c_1, c_2 \in I_\alpha$  (see below) and  $f$  is an isomorphism from  $B_\alpha \upharpoonright (1 - c_1)$  onto  $B_\alpha \upharpoonright (1 - c_2)$  then  $\theta > |\{x \in \text{atom}(B_\alpha) : x \leq_{B_\alpha} c_1 \text{ or } f(x) \neq x\}|$
- (e)  $I_\alpha = \{b \in B_\alpha : |\{x \in \text{atom}(B_\alpha) : x \leq b\}| < \theta\}$  is a maximal ideal of  $B_\alpha$
- (f) *there is an infinite set  $X$  of atoms of  $B_\alpha$  such that for every  $a \in B_\alpha, \{x \in X : x \leq a\}$  is a finite or co-finite subset of  $X$*
- (g) *if  $\alpha \neq \beta$  then for no  $a_\alpha \in I_\alpha, a_\beta \in I_\beta$  do we have  $B_\alpha \upharpoonright (1_{B_\alpha} - a_\alpha) \cong B_\beta \upharpoonright (1_{B_\beta} - a_\beta)$*
- (h)  $B^*$  is a superatomic Boolean Algebra
- (i)  $B^*$  has  $\mu$  atoms and  $\lambda$  elements<sup>1</sup>
- (j) *if  $\mathcal{U}$  is an infinite set of atoms of  $B^*$  then for some  $b \in B^*$  we have: [used?]*
  - (i)  $\{x \in \mathcal{U} : x \leq b\}$  is infinite
  - (ii)  $b/\text{id}_{\text{rk}(b, B^*)}(B^*)$  is an atom of  $B^*/\text{id}_{\text{rk}(b, B^*)}(B^*)$
  - (iii) *if  $b' < b$  &  $b' \in \text{id}_{\text{rk}(b, B^*)}(B^*)$  then  $\{x \in \mathcal{U} : x \leq b'\}$  is finite*
- (k) *if  $\lambda' > \mu$  then  $\chi, \mathcal{A}'$  satisfies:*
  - ( $\alpha$ )  $\mathcal{A}' \subseteq [\lambda']^{\aleph_0}$  is a MAD family of cardinality  $\leq \chi$  such that: every permutation  $\pi$  of  $\lambda' \setminus Z, Z \in [\lambda']^{<\sigma}$  satisfying  $A \in \mathcal{A}' \Rightarrow |A \Delta \pi''(A) \setminus Z| < \aleph_0$ , has support  $\{\alpha < \lambda' : \pi(\alpha) \neq \alpha\}$  of cardinality  $< \theta$
  - ( $\beta$ ) *for some ideal  $I^*$  of  $B^*$  containing  $\text{id}_1(B^*)$  the Boolean algebra  $B^*/I^*$  is isomorphic to  $\{a \subseteq \chi : a \text{ is finite or co-finite}\}$ .*

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<sup>1</sup>if there is a tree  $\mathcal{T}$  with  $\leq \mu$  nodes and  $\geq \lambda$  branches (= maximal linearly ordered subsets) then such  $B^*$  exists



Then we can find  $B$  such that:

- ( $\alpha$ )  $B$  is a superatomic Boolean Algebra
- ( $\beta$ )  $B$  has  $\lambda'$  atoms and  $\lambda$  elements
- ( $\gamma$ ) every automorphism  $g$  of  $B$  moves  $< \theta$  atoms; i.e.  
 $|\{x \in \text{atom}(B) : g(x) \neq x\}| < \theta$ .

*Proof.* Without loss of generality  $B^*$  is a Boolean Algebra of subsets of  $\mu$  with  $\{\{\alpha\} : \alpha < \mu\}$  being the atoms of  $B^*$ . If  $\lambda' = \mu$  let  $\mathcal{A}' = \emptyset$ .

Without loss of generality  $B_\alpha$  is a subalgebra of  $\mathcal{P}(X_\alpha)$  and the set of atoms of  $B_\alpha$  is  $\{\{x\} : x \in X_\alpha\}$ . Without loss of generality  $\alpha \neq \beta \Rightarrow X_\alpha \cap X_\beta = \emptyset$  and let  $X = \cup\{X_\alpha : \alpha < \mu\}$ .

Let  $Y^* \subseteq B^*$  be such that  $\{y/I^* : y \in Y^*\}$  is the set of atoms of  $B^*/I^*$  with no repetitions; without loss of generality for each  $y \in Y^*$  for some  $\alpha, y/\text{id}_\alpha(B^*)$  is an atom of  $B^*/\text{id}_\alpha(B^*)$  and  $(\forall z)[z \leq_{B^*} y \rightarrow z \in \text{id}_\alpha(B^*) \equiv z \in I^*]$  (just possibly decrease each  $y \in Y^*$ ).

Let  $Y$  be such that  $Y \subseteq B^*, \langle y/\text{id}_{\text{rk}(y, B^*)}(B^*) : y \in Y \rangle$  list with no repetitions  $\{y/\text{id}_{\text{rk}(y, B^*)}(B^*) : y/\text{id}_{\text{rk}(y, B^*)}(B^*) \text{ an atom of } B^*/\text{id}_{\text{rk}(y, B^*)}(B^*) \text{ and } \text{rk}(y, B^*) > 0\}$  and let  $D_y$  be the ultrafilter on  $B^*$  generated by  $\{y\} \cup \{1 - x : x \in B^*, \text{rk}(x, B^*) < \text{rk}(y, B^*)\}$  for each  $y \in Y$ . Without loss of generality  $Y^* \subseteq Y$  also clearly  $y \in Y \Rightarrow \{y' \in Y^* : y' - y \in \text{id}_{\text{rk}(y', B^*)}(B^*)\}$  is finite so without loss of generality is empty for  $y \in Y \setminus Y^*$  (singleton for  $y = Y^*$  of course), note that  $Y^*$  is of cardinality  $|\mathcal{A}'|$  and without loss of generality  $|Y \setminus Y^*| = \lambda$ .

First assume

$$\boxtimes_1 \ y \in Y \Rightarrow |y| = \mu.$$

Let  $g$  be a one-to-one function from  $\mu$  onto  $X$  and for  $A \in \mathcal{A}$  (from clause (b)) let  $\{\gamma_{A,k} : k < \omega\}$  list  $A$  without repetition. Let  $g^* : \mu \rightarrow \mu$  be  $g^*(\gamma) = \text{Min}\{\alpha < \mu : g(\gamma) \in X_\alpha\}$ . For each  $A \in \mathcal{A}$ , choose if possible an infinite  $u = u_A \subseteq \omega$  such that  $\langle g^*(\gamma_{A,k}) : k \in u \rangle$  is with no repetitions and  $\langle g^*(\gamma_{A,k}) : k \in u \rangle$  converge to some  $D_y, y = y_A \in Y$ . Note that the only case  $u$  is not well defined, is when the set  $\{g^*(\gamma_{A,k}) : k \in u\}$  is finite; we use clause (j) and properties of superatomic Boolean Algebras. As  $[y_A \text{ well defined} \Rightarrow |y_A| = \mu]$  by  $\boxtimes$ , clearly we can find  $\langle \alpha[A] : A \in \mathcal{A}, u_A \text{ well defined} \rangle$  such that:  $\alpha[A] < \mu, \alpha[A] \in y_A$  and  $(\forall z \in Y)[\alpha[A] \in z \Leftrightarrow z \in D_{y_A}]$  and  $\alpha[A_1] = \alpha[A_2] \Rightarrow A_1 = A_2$ . Let for  $\alpha < \mu, a_\alpha$  be  $\{g(\gamma_{A,k}) : k \in u_A\}$  if  $A \in \mathcal{A}, \alpha[A] = \alpha$  and  $u_A$  is well defined, and  $\emptyset$  otherwise. Toward defining our Boolean Algebra let  $\{x_\gamma^* : \gamma \in [\mu, \lambda']\}$  be pairwise distinct elements not in  $X$ . Let  $\mathcal{A}'' = \{\{\mu + i : i \in A\} : A \in \mathcal{A}'\}$  so it is a maximal almost

disjoint family of countable subsets of  $[\mu, \lambda')$ , as in clause (k) of the assumption so if  $\mu = \lambda'$  then  $\mathcal{A}'' = \emptyset = \mathcal{A}'$ ,  $\lambda' - \mu = 0$ ,  $(\lambda' - \mu)^{\aleph_0} = 0$ .

Now we define our Boolean Algebra  $\mathbf{B}$ . It is the Boolean Algebra of subsets of  $X^* = \bigcup_{\alpha < \mu} X_\alpha \cup \{x_\gamma^* : \gamma \in [\mu, \lambda')\}$  generated by the following:

- (i) the sets  $\{a \in B_\alpha : |a| < \theta\} \cup \{a \cup a_\alpha : a \in B_\alpha, |a| \geq \theta\}$  when  $\alpha < \mu$
- (ii)  $\{x_\gamma^*\}$  for  $\gamma \in [\mu, \lambda')$
- (iii) the sets  $c_y$  (for  $y \in Y$ ) where

$$c_y = \{x \in X : \text{for some } \alpha < \mu \text{ we have } x \in X_\alpha \text{ \& \ } \{\alpha\} \leq_{B^*} y\} \cup \{x_\gamma^* : \gamma \in [\mu, \lambda') \text{ and } y \in Y^* \text{ and } \gamma \in d_y\}.$$

Clearly

- ⊗<sub>1</sub>  $\mathbf{B}$  is a subalgebra of  $\mathcal{P}(X^*)$ , including all the singletons hence is atomic; has  $\lambda'$  atoms and  $\lambda$  elements
- ⊗<sub>2</sub> for  $\alpha < \mu$ , we have  $a \in B_\alpha$  &  $|a| < \theta \Rightarrow a \in \mathbf{B}$  &  $\mathbf{B} \restriction a = B_\alpha \restriction a$  but  $a \in B_\alpha \Rightarrow B_\alpha \restriction a$  is superatomic so  $\{a \in B_\alpha : |a| < \theta\} \subseteq \text{id}_\infty(\mathbf{B})$ .  
[Why? For the first implication we should check that every one of the generators of  $\mathbf{B}$  listed in (i), (ii), (iii) above satisfies: its intersection with  $a$  belong to  $B_\alpha \restriction a$ . The rest follows.]
- ⊗<sub>3</sub> for  $\alpha < \mu$ ,  $\{a \in \mathbf{B} : a \subseteq X_\alpha \cup a_\alpha : |a| < \theta\}$  satisfies
  - (i) it is equal to  $\{a \cup b : a \in B_\alpha \text{ \& \ } |a| < \theta \text{ and } b \subseteq a_\alpha \text{ is finite}\}$
  - (ii) it is a maximal ideal of  $\mathbf{B} \restriction (X_\alpha \cup a_\alpha)$
  - (iii) it is included in  $\text{id}_\infty(\mathbf{B})$ .  
[Why? Just think.]
- ⊗<sub>4</sub>  $\alpha < \mu \Rightarrow X_\alpha \cup a_\alpha \in \text{id}_\infty(\mathbf{B})$   
[Why? First  $X_\alpha \cup a_\alpha \in \mathbf{B}$  by clause (i) above, second if  $X_\alpha \cup a_\alpha \notin \text{id}_\infty(\mathbf{B})$  then by ⊗<sub>3</sub> above  $(X_\alpha \cup a_\alpha)$  is an atom of  $\mathbf{B}/\text{id}_\infty(\mathbf{B})$  for  $\alpha$  large enough, hence  $X_\alpha \cup a_\alpha$  belong to  $\text{id}_{\alpha+1}(\mathbf{B})$ , contradiction
- ⊗<sub>5</sub> for  $\alpha < \mu$ ,  $\mathbf{B} \restriction (X_\alpha \cup a_\alpha) \cong B_\alpha$  hence if  $\alpha < \beta < \omega$  then for no  $c_\alpha$  such that  $c_\alpha \in B_\alpha, c_\alpha \leq X_\alpha \cup a_\alpha, |c_\alpha| < \theta$  and  $c_\beta \in B_\beta, c_\beta \leq X_\beta \cup a_\beta, |c_\beta| < \theta$  do we have  $\mathbf{B} \restriction (X_\alpha \cup a_\alpha \setminus c_\alpha) \cong \mathbf{B} \restriction (X_\beta \cup a_\beta \setminus c_\beta)$ .  
[Why? By clauses (f) + (e) of the assumption, the “hence” follows by clause (g) of the assumption.]

Let  $I_1$  be the ideal  $[X^*]^{<\theta} \cap \mathbf{B}$  of  $\mathbf{B}$ . So clearly

$$\otimes_6 I_1 \subseteq \text{id}_\infty(\mathbf{B}).$$

We shall prove that

$$\otimes_7 \mathbf{B}/I_1 \text{ is isomorphic to a homomorphic image of } B^*.$$

Toward proving  $\otimes_7$  let  $S = \{x_\gamma^* : \gamma \in [\mu, \lambda']\}$  and define a function  $h$  as follows: its domain is  $\{c_y : y \in Y\} \cup \{X_\alpha \cup a_\alpha : \alpha < \mu\}$  and  $h(c_y) = y, h(X_\alpha \cup a_\alpha) = \{\alpha\}$ . Now

- (\*)<sub>0</sub>  $(X_\alpha \cup a_\alpha)/I_0$  is an atom of  $\mathbf{B}/I_1$   
[why? by  $\otimes_3$ .]
- (\*)<sub>1</sub>  $\{b/I_1 : b \in \text{Dom}(h)\}$  is a subset of  $\mathbf{B}/I_1$  which generates it  
[why? see the definitions of  $\mathbf{B}$  and of  $I_1$ .]
- (\*)<sub>2</sub> if  $n_1 \leq n < \omega, m_1 \leq m < \omega, y_0, \dots, y_{n-1} \in Y$  is with no repetitions,  
 $\alpha_0, \dots, \alpha_{m-1} < \mu$  is with no repetitions, then:

in  $\mathbf{B}, \tau_1 =: \bigcap_{\ell < n_1} c_{y_\ell} \cup \bigcap_{\ell < m_1} (X_{\alpha_\ell} \cup a_{\alpha_\ell}) - \bigcup_{\ell=n_1}^{n-1} c_{y_\ell} \cup \bigcup_{\ell=m_1}^{m-1} (X_{\alpha_\ell} \cup a_{\alpha_\ell})$  belongs  
to  $I_1$  iff

in  $B^*, \tau_2 =: \bigcap_{\ell < n_1} y_\ell \cup \bigcap_{\ell < m_1} \{\alpha_\ell\} - \bigcup_{\ell=n_1}^{n-1} y_\ell \cup \bigcup_{\ell=m_1}^{m-1} \{\alpha_\ell\}$  is empty.

[Why? First, assume that the second statement holds then trivially  $\tau'_1 =:$

$$\bigcap_{\ell < n_1} (c_{y_\ell} \setminus S) \cup \bigcap_{\ell < m_1} X_{\alpha_\ell} - \bigcup_{\ell=n_1}^{n-1} (c_{y_\ell} \setminus S) \cup \bigcup_{\ell=m_1}^{m-1} X_{\alpha_\ell} = \bigcup \{X_\beta : B^* \models \{\alpha\} \leq \tau_2\} = \emptyset \text{ but } \tau'_1 \Delta \tau_1 \subseteq S \cup \bigcup_{\ell < m} a_{\alpha_\ell} \text{ but } a_{\alpha_\ell} \in I_0 \subseteq [X^*]^{<\theta} \text{ and } \tau'_1 = \emptyset, \text{ so}$$

$$\tau_1 \subseteq S \text{ mod } [X^*]^{<\theta}.$$

Now if  $\tau_1 \cap S \notin J_0$  then  $\tau_1 \cap S$  is infinite, clearly  $\lambda' > \mu$ , so as  $\{d_z : z \in Y^*\}$  is a MAD family of subsets of  $\lambda' \setminus \mu$ , necessarily for some  $z \in Y^*$  we have  $\tau_1 \cap S \cap d_z$  is infinite. As  $\tau_1 \cap S \cap d_z \subseteq c_{y_\ell}$  for  $\ell < n_1$ , necessarily  $y_\ell = z$ , hence  $y_0 = z, n_2 = 1$ . Similarly  $\ell \in [n_1, n_2) \Rightarrow y_\ell \neq z$  hence  $\ell \in [n_1, n) \Rightarrow y_\ell \cap y_0 = y_\ell \cap z \in \text{id}_{\text{rk}(z, B^*)}(B^*) \Rightarrow |d_z \cap c_{y_\ell}| < \aleph_0$ . Hence clearly  $\ell \in [n_1, n) \Rightarrow y_\ell \notin D_0$  but  $y_0 \in D_z$  and  $\alpha < \mu \Rightarrow \{\alpha\} \notin D_z$  (as  $y \in Y$ !) hence  $B^* \models \text{"}\tau_2 > 0\text{"}$ , contradiction, so necessarily  $\tau_1 \cap S$  is finite hence  $\in I_1$ .

Second, if the second statement fails, then for some  $\beta < \mu, B^* \models \{\beta\} \leq \tau_2$ , but then  $X_\beta \subseteq \tau'_1$  and as above  $\tau'_1 \supseteq \tau_1 \setminus S \supseteq X_\beta \text{ mod } I_1$  but  $S \cap X_\beta = \emptyset$ , so  $X_\beta \subseteq \tau_1 \text{ mod } I_1$ ; now  $X_\beta \notin I_0$  (as  $|X_\beta| \geq \theta$  by clause (e) of the assumptions) hence  $\tau_1 \notin I_1$ . So we have proved (\*)<sub>2</sub>.]

Now by (\*),  $\otimes_7$  follows, in fact  $h$  induces an isomorphism  $\hat{h}$  from  $\mathbf{B}/I_0$  onto  $B^*$ . But  $B^*$  is superatomic and  $I_0 \subseteq \text{id}_\infty(\mathbf{B})$  by  $\otimes_6$  hence

$\otimes_8$   $\mathbf{B}$  is superatomic.

For the rest of the proof let  $f \in \text{AUT}(\mathbf{B})$  and toward contradiction we assume  $\text{sup}(f) = \{x \in \text{atom}(\mathbf{B}) : f(x) \neq x\}$  has cardinality  $\geq \theta$ .

Recall that  $I_1 = \{a \in \mathbf{B} : |a| < \theta\}$  so necessarily  $f$  maps  $I_1$  onto itself. Note that  $\{X_\alpha \cup a_\alpha / I_1 : \alpha < \mu\} / I_1$  list the atoms of  $\mathbf{B}/I_1$ . Assume  $f(X_\alpha \cup a_\alpha) / I_1 = (X_\beta \cup a_\beta) / I_1$ ,  $\alpha \neq \beta$ ; let  $c_1 = (X_\alpha \cup a_\alpha) - f^{-1}(X_\beta \cup a_\beta)$  and  $c_2 = (X_\beta \cup a_\beta) - f(X_\alpha \cup a_\alpha)$ , so both being the difference of two members of  $\mathbf{B}$  are in  $\mathbf{B}$  and  $c_1 \leq X_\alpha \cup a_\alpha$ ,  $c_2 \leq X_\beta \cup a_\beta$  hence  $c_1 \in B_\alpha$ ,  $c_2 \in B_\alpha$ . Clearly  $f \upharpoonright (\mathbf{B} \upharpoonright (X_\alpha \cup a_\alpha - c_1))$  is an isomorphism from  $\mathbf{B} \upharpoonright (X_\alpha \cup a_\alpha - c_1)$  onto  $\mathbf{B} \upharpoonright (X_\beta \cup a_\beta - c_2)$ , contradicting  $\otimes_5$ . Hence the automorphism  $f$  induced on  $B^*/I_1$  maps each atom to itself hence is the identity. Also for  $\alpha < \mu$  we have  $(X_\alpha \cup a_\alpha) \Delta f(X_\alpha \cup a_\alpha) \in I_1$ , that is, has cardinality  $< \theta$ . So for each  $\alpha < \mu$ , letting  $c_\alpha^1 = (X_\alpha \cup a_\alpha) - f^{-1}(X_\alpha \cup a_\alpha) \in I_\alpha$  and  $c_\alpha^2 = (X_\alpha \cup a_\alpha) - f(X_\alpha \cup a_\alpha) \in I_\alpha$ ,  $f \upharpoonright (B_\alpha \upharpoonright (1 - c_\alpha^1))$  is an isomorphism from  $B_\alpha \upharpoonright (1 - c_\alpha^2)$  onto  $B_\alpha \upharpoonright (1 - c_\alpha^2)$  hence

$\boxtimes_2$   $Z_\alpha = \{x : x \text{ an atom of } B_\alpha, x \leq_{B_\alpha} c_\alpha^1 \vee f(x) \neq x\}$  has cardinality  $< \theta$

by clause (d) of the assumptions on  $B_\alpha$ . Let  $v =: \{\alpha < \mu : \text{for some } x \in X_\alpha \text{ we have } f(\{x\}) \neq \{x\}\}$ . Assume for the time being

$\boxtimes_3$   $v$  has cardinality  $\geq \text{cf}(\theta)$ .

For  $\alpha \in v$  choose  $x_\alpha \in X_\alpha$  such that  $f(x_\alpha) \neq x_\alpha$  and shrinking  $v$  without loss of generality  $\alpha, \beta \in v \Rightarrow x_\alpha \neq f(x_\beta)$ . Let  $g : v \rightarrow \mu + 1$  be such that  $f(x_\alpha) \subseteq X_{g(\alpha)}$  where we stipulate  $X_\mu = S$ . Applying the above to  $f^{-1}$  without loss of generality either  $g$  is one-to-one into  $\mu$  or  $g$  is constantly  $\mu$ . Hence by clause (b) of the assumption for some  $A \in \mathcal{A}$  we have  $(\forall \gamma \in A)[\{\gamma\} \in \{x_\alpha : \alpha \in v\}]$ . So  $\alpha[A] < \mu$  is well defined and an easy contradiction. We can conclude that  $\neg \boxtimes_3$  hence  $Z =: \{x \in X : f(\{x\}) \neq \{x\}\}$  has cardinality  $< \text{cf}(\theta)$  hence  $|\{x \in X : f(x) \neq x\}| < \theta$ . If  $\mu = \lambda'$  we are done so assume  $\mu < \lambda'$ .

Now  $S = \{x_\gamma^* : \gamma \in [\mu, \lambda')\} = X^* \setminus X \subseteq X^*$  satisfies:

$\otimes_9(\alpha)$   $(\forall b \in \mathbf{B})(b \cap S \text{ infinite} \Rightarrow 1 \leq \text{rk}(b/I_1, \mathbf{B}/I_1))$  and

( $\beta$ ) if  $S'$  satisfies the property in clause ( $\alpha$ ), then  $|S' \setminus S| < \sigma$

[Why? Clause ( $\alpha$ ) is proved by inspecting the definition of  $\mathbf{B}$ . As for clause ( $\beta$ ), if  $|S' \setminus S| \geq \sigma$  as  $S' \setminus S \subseteq X$  clearly then there is  $A \in \mathcal{A}$  such that  $\{g(i) : i \in A\} \subseteq S' \setminus S$ . First if  $\alpha =: \alpha[A]$  is well defined then  $X_\alpha \cup a_\alpha \in \mathbf{B}$ ,  $\text{rk}((X_\alpha \cup a_\alpha)/I_1, \mathbf{B}/I_1) = 0 < 1$  but  $(X_\alpha \cup a_\alpha) \cap S' \supseteq a_\alpha$  is infinite;

contradiction. Second if  $\alpha[A]$  is not well defined then for some  $\alpha < \mu$  we have  $\{g(i) : i \in A\} \cap X_\alpha$  is infinite and we get a similar contradiction.]

Hence  $S_f^* =: \{x_\gamma^* : \gamma \in [\mu, \lambda') \text{ and } f^n(\{x_\gamma^*\}) \subseteq X \text{ or } f^{-n}(\{x_\gamma^*\}) \subseteq X \text{ for some } n < \omega\}$  has cardinality  $< \sigma$  (this also follows from the previous paragraph recalling  $\sigma = \text{cf}(\sigma) > \aleph_0$ ).

Also for  $y \in Y^*$  letting  $\gamma = \text{rk}(y, B^*)$  we have  $c_y \Delta f(c_y) \in I_1$ , (just recall that the automorphism  $f$  induced on  $\mathbf{B}/I_1$  is the identity, and recall that  $[d \subseteq S \ \& \ d \in I_1 \Rightarrow d \text{ is finite}]$ , hence the symmetric difference of  $\{\{x_\gamma^*\} : \gamma \in d_y\} \setminus S_f^*, \{f\{x_\gamma^*\} : \gamma \in d_y\} \setminus S_f^*$  is finite.

As  $\{d_y : y \in Y^*\}$  is MAD as in clause the set  $\{\gamma \in [\mu, \lambda') : f(\{x_\gamma^*\}) \neq \{x_\gamma^*\}\}$  is of cardinality  $< \theta$ ; so seemingly we are done.

Not exactly: we have assumed  $\boxtimes$ , i.e.  $y \in Y \Rightarrow |y| \geq \mu$ .

To eliminate this we make some minor changes. First without loss of generality  $B^*$  is a Boolean Algebra of subsets of  $\{\alpha : \alpha < \mu \text{ even}\}$  with the singletons being its atoms. Second, for  $A \in \mathcal{A}$ , if possible we choose  $u = u_A$  as follows:

- (a) either  $(\alpha)$  or  $(\beta)$  where
  - $(\alpha)$   $g^*(\gamma_{A,k})$  is odd for every  $k \in u$
  - $(\beta)$   $g^*(\gamma_{A,k})$  is even for every  $k \in u$
- (b)  $\langle g^*(\gamma_{A,k}) : k \in u \rangle$  is with no repetitions
- (c) if case  $(\beta)$  occurs in  $A$ , then there is a unique  $y = y_A \in Y$  such that  $\langle \{g^*(\gamma_{A,k})\} : k \in u \rangle$  converge to  $D_{y_A}$ .

Note

- (\*) if  $u_A$  is not well defined then for some finite  $w \subseteq \mu$  we have
 
$$\{g(\gamma_{A,k}) : k < \omega\} \subseteq \bigcup_{\alpha \in \omega} X_\alpha.$$

Now we choose  $\langle \alpha[A] : A \in \mathcal{A}, u_A \text{ will define } \rangle$  such that:

- (\*\*)  $\langle \alpha[A] : A \in \mathcal{A}, u_A \text{ well defined } \rangle$  list with no repetitions the odd ordinals  $< \mu$ .

We define  $a_\alpha$ , etc. as before.

Lastly, defining  $c_y$ , we add  $\{x : \text{for some } A \in \mathcal{A}, \alpha[A] \text{ is well defined, } x \in X_{\alpha[A]} \text{ and } B^* \models y_A \leq y\}$ .

Note that we just replace  $B^*$  by  $B^{**} \subseteq \mathcal{P}(\mu)$  where without loss of generality  $B^* \subseteq$

$\mathcal{P}(\mu)$ ,  $\text{atom}(B^*) = \{\{\alpha\} : \alpha < \mu\}$ , let  $f : \mu \rightarrow \mu$  be onto such that  $(\forall \alpha)(\exists^\mu \beta)[f(\beta) = \alpha]$  and we let  $B^{**}$  be the subalgebra of  $\mathbf{P}(\mu)$  generated by  $\{\{\alpha\} : \alpha < \mu\} \cup \{\{\beta < \mu : f(\beta) \in y\} : y \in B^*\}$ .  $\square_{2.1}$

Discussion:

Why do we use MAD families  $\mathcal{A} \subseteq [\mu]^{\aleph_0}$  and not  $\subseteq [\mu]^{\aleph_1}$ ? If we use the latter, we have to take more care about superatomicity as the intersections of such members may otherwise contradict superatomicity.

§3 SUFFICIENT CONDITIONS FOR THE EXISTENCE OF  $\langle B_i : i < \mu \rangle$ 

Here we shall show that the assumptions of 1.1 are reasonable. Now in 3.2 we shall reduce the clause (k) of 1.1 to  $\text{Pr}(\lambda', \theta)$  where  $\text{Pr}$  formalizes clause (b) there. In 3.3, 3.5 we give sufficient conditions for  $\text{Pr}(\mu, \sigma)$ . In fact, it is clear that (high enough) it is not easy to fail it. In 3.9 we give a sufficient condition for a strong version of clauses (e) - (f) of 1.1 (and earlier deal with the conditions appearing in it). So for not having the assumptions of 1.1 it has large consistency strength.

**3.1 Definition.** 1)  $\text{Pr}(\chi, \mu, \sigma)$  means that for some  $\mathcal{A}$  we have:

- (a)  $\mathcal{A} \subseteq [\mu]^{\aleph_0}$
- (b)  $\mathcal{A}$  is almost disjoint, i.e.  $A \neq B \in \mathcal{A} \Rightarrow |A \cap B| < \aleph_0$
- (c)  $|\mathcal{A}| = \chi$
- (d)  $(\forall A \in [\mu]^\sigma)(\exists A \in \mathcal{A})[A \subseteq^* A]$ .

2) If we omit  $\chi$  we mean “some  $\chi$ ”.

3) We call  $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$  saturated if for every  $A \in [\lambda]^{\aleph_0}$  not almost contained in a finite union of members of  $\mathcal{A}$ , almost contains a member of  $\mathcal{A}$ .

**3.2 Fact:** 1) Clause (b) of the assumption of 2.1 is equivalent to  $\text{Pr}(\mu, \mu, \sigma)$ .

2) Clause (k)( $\alpha$ ) of the assumption of 2.1 follows from  $\text{Pr}(\chi, \lambda', \sigma)$  &  $\chi \geq 2^{\aleph_0}$ .

3) If  $\mathcal{A} \subseteq [\mu]^{\aleph_0}$  is almost disjoint and is saturated then  $\text{Pr}(|\mathcal{A}|, \mu, \aleph_1)$ .

4) If  $\mu = \mu^{\aleph_0} \geq \sigma$  then  $\text{Pr}(\mu, \sigma) \equiv \text{Pr}(\mu, \mu, \sigma)$  and  $\chi \neq \mu \Rightarrow \neg \text{Pr}(\chi, \mu, \sigma)$ .

5) For any  $\lambda' \geq \aleph_0$  there is a MAD family  $\mathcal{A} \subseteq [\lambda']^{\aleph_0}$  of cardinality  $[\lambda']^{\aleph_0}$  satisfying clause (k)( $\alpha$ ) of 2.1.

*Proof.* 1) Read the two statements.

2) Let  $\mathcal{A} \subseteq [\lambda']^{\aleph_0}$  exemplify  $\text{Pr}(\lambda', \sigma)$ . For each  $A \in \mathcal{A}$  we can find  $\langle B_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$  such that:

- (i)  $B_{A,\zeta} \in [A]^{\aleph_0}$
- (ii)  $\zeta \neq \varepsilon \Rightarrow B_{A,\zeta} \cap B_{A,\varepsilon}$  is finite
- (iii) if  $\pi$  is a partial one-to-one function from  $A$  to  $A$  such that  $x \in \text{Dom}(\pi) \rightarrow x \neq \pi(x)$  then for some  $\zeta < 2^{\aleph_0}$  we have  $\alpha \in B_{A,\zeta} \Rightarrow \alpha \notin \text{Dom}(\pi) \vee \pi(\alpha) \notin B_{A,\zeta}$ .

Why? First find  $\langle B'_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$  satisfying (i), (ii), let  $\langle \pi_\zeta : \zeta < 2^{\aleph_0} \rangle$  list the  $\pi$ 's from (iii) and chose  $B_{A,\zeta} \in [B'_{A,\zeta}]^{\aleph_0}$  to satisfy clause (iii) for  $\pi_\zeta$ .

Lastly,  $\mathcal{A}' = \{B_{A,\zeta} : A \in \mathcal{A} \text{ and } \zeta < 2^{\aleph_0}\}$  it is as required.

3), 4) Easy.

5) Starting with AD  $\mathcal{A}_0 \subseteq [\lambda']^{\aleph_0}$  of cardinality  $9\lambda'$ , extend it to a MAD one  $\mathcal{A}_1$  and then apply the proof of part (2).  $\square_{3.2}$

**3.3 Claim.** 1) Assume

- (a)  $\kappa_n < \kappa_{n+1} < \kappa < \mu_n < \mu_{n+1} < \mu$  for  $n < \omega$
- (b)  $\kappa = \sum \kappa_n, \mu = \sum \mu_n$  and  $\max \text{pcf} \{ \kappa_n : n < \mu \} > \mu$
- (c)  $\kappa$  strong limit and  $2^\kappa \geq \mu^+$
- (d)  $\langle \mu_n : n < \omega \rangle$  satisfies the requirements from [Sh 513, §1] or at least the conclusion.

Then for every  $\lambda \geq \kappa$  we can find  $\{\bar{A}_\alpha : \alpha < \alpha^*\}$  such that

- ( $\alpha$ ) each  $\bar{A}_\alpha$  has the form  $\langle A_{\alpha,n} : n < \omega \rangle$ , it belongs to  $\prod_{n < \omega} [\lambda]^{\kappa_n}$  and for each  $\alpha$  we have  $\langle A_{\alpha,n} : n < \omega \rangle$  pairwise disjoint
- ( $\beta$ ) if  $\alpha \neq \beta$ , then  $\bar{A}_\alpha, A_\alpha$  are almost disjoint which means  $f \in \prod_{n < \omega} A_{\alpha,n}$  &  $f' \in \prod_{n < \omega} A_{\beta,n} \Rightarrow |\text{Rang}(f) \cap \text{Rang}(f')| < \aleph_0$
- ( $\gamma$ ) if  $\bar{A} \in \prod_{n < \omega} [\lambda]^{\kappa_n}$  then for some  $\alpha < \alpha^*$  and one to one function  $h_1, h_2 \in {}^\omega \omega$  we have  $\kappa = \lim \langle |A_{h_1(n)} \cap A_{\alpha, h_2(n)}| : n < \omega \rangle$ .

2) We can conclude in (1) that: there is  $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$ , an almost disjoint family such that  $(\forall B \in [\lambda]^\kappa)(\exists A \in \mathcal{A})(A \subseteq B)$ .

3) If in part (1) instead (a)-(d) we just assume  $\kappa$  strong limit of cofinality  $\aleph_0$ , and SCH +  $(\forall \lambda > \kappa)[\text{cf}(\lambda) = \aleph_0 \rightarrow \diamond_{\lambda^+}]$ , then the conclusion of (1) holds.

*Proof.* By [Sh 460], [Sh 668, §3] (even more).

**3.4 Remark.** Are the hypotheses of 3.3(1) reasonable?

1) If for some strong limit  $\kappa$  of cofinality  $\aleph_0$ ,  $2^\kappa > \kappa^{+\omega}$ , then we can let  $\mu_n = \kappa^{+1+n}$  and  $\langle \kappa_n : n < \omega \rangle$  as in clause (a), (b), (c) there exists (by [Sh:g, Ch.IX, §5], and it is hard not to satisfy clause (d) (see [Sh 513]).

2) Clause (c), i.e.  $\kappa$  strong limit, is needed just to start the induction. If  $\kappa = \aleph_\omega \leq 2^{\aleph_0}$  we have a similar theorem.

3) If  $\aleph_\omega \leq 2^{\aleph_0}$  and  $\aleph_\omega$  is as required in [Sh 513, §1] then we have a parallel theorem.

We quote [GJSh 399] in 3.5(1) and (2) is immediate starting the induction with the known  $\text{Pr}(\lambda, \aleph_1)$  for  $\aleph_0 < \lambda \leq 2^{\aleph_0}$ .



**3.5 Claim.** 1) Assume  $CH + SCH + (\forall \mu)(\text{cf}(\mu) = \aleph_0 < \mu \rightarrow \square_{\mu^+})$ . Then there is a saturated MAD family  $\mathcal{A}_\lambda \subseteq [\lambda]^{\aleph_0}$  for every uncountable  $\lambda$ .  
 2) If  $SCH + (\forall \mu)[\text{cf}(\mu) = \aleph_0 \rightarrow \square_{\mu^+}]$ , then  $\text{Pr}(\lambda, \aleph_1)$  for every  $\lambda > \aleph_0$ .

**3.6 Definition.** Let  $\mu \geq \theta$ .

1) Let  $\mathcal{S}_\theta$  be the class of  $\bar{a} = \langle a_n : n < \omega \rangle$  such that  $|a_n| \leq \theta$ ,  $\text{cf}(\theta) = \aleph_0 \Rightarrow |a_n| < \theta$  and  $\theta = \limsup_n |a_n|$ . Let  $\mathcal{S}_{\theta, \mu} =: \{\bar{a} : \bar{a} = \langle a_n : n < \omega \rangle, a_n \in [\mu]^{\leq \theta}, a_n \subseteq a_{n+1} \text{ and } \theta = \limsup_{n < \omega} |a_n|\}$ .

3) For  $\bar{a} \in \mathcal{S}_\theta$  let  $\text{set}(\bar{a}) = \{w : |w| = \aleph_0 \text{ and } w \subseteq \bigcup_{n < \omega} a_n \text{ and } n < \omega \Rightarrow |w \cap$

$$a_n \setminus \bigcup_{\ell < n} a_\ell| < \aleph_0\}.$$

4) For  $\bar{a}, \bar{b} \in \mathcal{S}_\theta$  let  $\bar{a} \leq^* \bar{b}$  mean  $\text{set}(\bar{a}) \supseteq \text{set}(\bar{b})$ .

5) We say  $\bar{a}, \bar{b} \in \mathcal{S}_\theta$  are compatible if  $(\exists \bar{c} \in \mathcal{S}_\theta)[\bar{a} \leq^* \bar{c} \ \& \ \bar{b} \leq^* \bar{c} \ \& \ \bigcup_n c_n \subseteq \bigcup_n a_n \cap \bigcup_n b_n]$ .

**3.7 Definition.** Let  $\boxtimes_{\theta, \mu}$  be

$\boxtimes_{\theta, \mu}$  there is  $\mathcal{S}^* \subseteq \mathcal{S}_{\theta, \mu}$  such that:

- (a) for every  $\bar{a} \in \mathcal{S}$  there is  $\bar{b} \in \mathcal{S}^*$  compatible with  $\bar{a}$
- (b) if  $\bar{a} \neq \bar{b} \in \mathcal{S}^*$  then the  $\text{set}(\bar{a}) \cap \text{set}(\bar{b}) = \emptyset$ .

**3.8 Claim.** If  $\theta$  is strong limit,  $\theta > \text{cf}(\theta) = \aleph_0$  and  $\mu \in (\theta, (2^\theta)^{+\omega})$  satisfies  $\boxtimes_{\theta, \mu}$  below then  $\boxtimes_{\theta, \mu}$  from 3.7 holds where:

$$\boxtimes_{\theta, \mu} \ \mu = 2^\theta, \text{pp}_{J_\omega^{\text{bd}}}(\theta) = 2^\theta.$$

*Proof.* Straight. First assume  $\mu \leq 2^\theta$  as  $\mathcal{S}_{\theta, \mu} = \mu^\theta = 2^\theta$ , we can find  $\langle \bar{a}^\alpha : \alpha < 2^\theta \rangle$  listing  $\mathcal{S}_{\theta, \mu}$ . Now we choose  $\gamma_0(\alpha), \bar{b}^\alpha$  by induction on  $\alpha < 2^\theta$  such that

- (a)  $\bar{b}^\alpha \in \mathcal{S}_{\theta, \mu}$
- (b)  $\beta < \alpha \Rightarrow \text{set}(\bar{a}) \cap \text{set}(\bar{b}) = \emptyset$
- (c)  $\bar{a}^{\gamma(\alpha)}, \bar{b}^\alpha$  are compatible
- (d)  $\gamma(\alpha) = \text{Min}\{\gamma : \bar{a}^\gamma \text{ incompatible with } \bar{b}^\beta \text{ for every } \beta < \alpha\}$ .

Arriving to  $\alpha$  choose  $\gamma(\alpha)$  by clause (d), we can find  $\kappa_n = \text{cf}(\kappa_n) < \theta$  such that  $\prod_{n < \omega} \kappa_n / J_\omega^{\text{bd}}$  has true cofinality  $> |\alpha|$ . Let  $h : \omega \rightarrow \omega$  be increasing such that  $|b_{k(n)}^{\gamma(\alpha)}| \geq \kappa_n$  choose  $\gamma_{\alpha, \zeta, n} \in b_{h^n}^{\gamma(\alpha)}$  for  $\zeta < \kappa_n$  increasing with  $\zeta$ . For each  $\beta < \alpha$  define  $g_{\beta, \alpha} \in \prod_n (\kappa_n^{+1})$  by  $g_{\beta, \alpha}(n) = \sup\{\zeta : \gamma_{\alpha, \zeta, n} \in \bigcup_{m < \omega} a_m^\beta\}$ . Easily  $g_{\beta, \alpha} <_{J^{\text{bd}} \omega}$   $\langle \kappa_n : n < \omega \rangle$  hence there is  $\langle \zeta_n : n < \omega \rangle$  such that  $(\forall \beta < \alpha)[g_{\beta, \alpha} <_{J^{\text{bd}} \omega} \langle \zeta_n : n < \omega \rangle]$  and let  $b_n^\alpha = \{\gamma_{\alpha, \zeta, n} : n < \omega \text{ and } \zeta \in [\zeta_n, \kappa_n)\}$ . Easy to carry and give the conclusion. For  $\mu = (2^\theta)^{+n}$ , use induction on  $n$  (as in [Sh 668, §3] or [EH:1, p.xx]. ? EH:1 ?

**3.9 Claim.** 1) Assume  $\theta$  is strong limit,  $\aleph_0 = \text{cf}(\theta) < \theta$ . Assume further  $\theta \leq \kappa \leq 2^{2^\theta}$ ,  $\mu = 2^\kappa$  and  $\boxtimes_{\theta, \kappa}$  (from 3.7) holds and  $\mu = \mu^{\aleph_0}$ . Then some  $\bar{B} = \langle B_\alpha : \alpha < \mu \rangle$  satisfies clauses (b) - (g) of 2.1; in fact  $B_\alpha$  is a subalgebra of  $\mathcal{P}(2^\theta)$  with 2 levels and  $\text{id}_{<\infty}(B_\alpha)$  is included in  $\{a \subseteq 2^\theta : a \text{ countable or co-countable}\}$ .  
2) As above except that  $2^\theta = \theta^{\aleph_0}$ ,  $\theta > \text{cf}(\theta) = \aleph_0$ .

*Proof.* 1) Let for simplicity  $\theta = \sum \theta_n$ ,  $\theta_n < \theta_{n+1} < \theta$ .

Fact: Letting  $\bar{a}^* = \langle \theta_n : n < \omega \rangle$ , i.e.  $a_n^* = \theta_n$  we can find  $\bar{t}^a = \langle t_{\ell, \alpha} : \ell < 3, \alpha < 2^\theta \rangle$  such that:

- (i)  $t_{\ell, \alpha} \in \text{set}(\bar{a}^*)$  has order type  $\omega$
- (ii) for some one to one onto  $\pi : 2^\theta \times 2^\theta \rightarrow 2^\theta$  we write  $t_{2, \alpha, \beta}$  for  $t_{2, \pi(\alpha, \beta)}$
- (iii) if  $(\ell_1, \alpha_1) \neq (\ell_2, \alpha_2)$  then  $t_{\ell_1, \alpha_1} \cap t_{\ell_2, \alpha_2}$  is finite
- (iv) if  $\bar{a} \in \mathcal{S}_{\theta, \kappa}$  and  $\bigcup_{n < \omega} a_n \subseteq \theta$  then for some  $\alpha < 2^\theta$  we have  $\beta < 2^\theta \Rightarrow t_{2, \alpha, \beta} \in \text{set}(\bar{a})$
- (v) if  $\bar{a}, \bar{b} \in \mathcal{S}_{\theta, \kappa}$  and  $\bigcup_{n < \omega} a_n \cup \bigcup_{n < \omega} b_n \subseteq \theta$  and  $\text{set}(\bar{a}) \cap \text{set}(\bar{b}) = \emptyset$  and  $h : \bigcup_{n < \omega} a_n \rightarrow \bigcup_{n < \omega} b_n$  is one to one and maps  $a_n$  onto  $b_n$  then for some  $\alpha$ ,  $t_{0, \alpha} \in \text{set}(\bar{a})$ ,  $t_{1, \alpha} \in \text{set}(\bar{b})$  and  $h$  maps  $t_{0, \alpha}$  into a co-infinite subset of  $t_{1, \alpha}$ .

*Proof of the fact.* Straight.

Construction: Let  $\mathcal{S}^* = \{\bar{a}^\gamma : \gamma < \kappa\}$  exemplify  $\boxtimes_{\theta, \kappa}$ , without loss of generality  $\bar{a} \in \mathcal{S}^* \Rightarrow \bigwedge_{n < \omega} |a_n| = \theta_n$ ; (because for every  $\bar{a} \in \mathcal{S}_\theta$  there is  $\bar{a}' \in \mathcal{S}_\theta$ ,  $|a'_n| = \theta_n$  and

$\text{set}(\bar{a}') = \text{set}(\bar{a})$ ). Let  $\{X_\gamma : \gamma < \kappa\}$  be a sequence of subsets of  $2^\theta$  such that  $\gamma_1 \neq \gamma_2 \Rightarrow |X_{\gamma_1} \setminus X_{\gamma_2}| = 2^\theta$ ; let  $\langle Y_j : j < \mu \rangle$  be a sequence of subsets of  $\kappa$  such that  $j_1 \neq j_2 \Rightarrow |Y_{j_1} \setminus Y_{j_2}| = \kappa$ , let  $g_\gamma$  be a one to one mapping from  $\theta$  into  $\bigcup_{n < \omega} a_n^\gamma$  mapping  $\theta_n$  onto  $a_n^\gamma$ , and lastly let  $t_{\ell, \alpha}^\gamma = g_\gamma''(t_{\ell, \alpha})$  and  $t_{\ell, \alpha, \beta}^\gamma = g_\gamma''(t_{\ell, \alpha, \beta}^\gamma)$ . Let  $t_{3, \alpha, \beta}^\gamma = \{g_\gamma(\varepsilon) : \varepsilon \in t_{2, \alpha, \beta} \text{ and } |t_{2, \alpha, \beta} \cap \varepsilon| \text{ is even}\}$ . Let  $t_{\zeta, \alpha, \beta}^\gamma = \{\zeta \in t_{2, \alpha, \beta}^\gamma : |t_{2, \alpha, \beta}^\gamma \cap \zeta| \text{ is even}\}$ .

For  $j < \mu$ , let  $\mathcal{A}_j$  be the following family of subsets of  $\kappa$

$$t_{0, \alpha}^\gamma, t_{1, \alpha}^\gamma \text{ when } \gamma < \kappa, \alpha < 2^\theta$$

$$t_{2, \alpha, 1+\beta}^\gamma \text{ when } \beta \notin X_\gamma, \alpha < 2^\theta$$

$$t_{3, \alpha, 1+\beta}^\gamma \text{ when } \beta \in X_\gamma, \alpha < 2^\theta$$

$$t_{2, \alpha, 0}^\gamma \text{ when } \gamma \notin Y_j \text{ and } t_{3, \alpha, 0}^\gamma \text{ when } \gamma \in Y_j.$$

Clearly

$$\odot_1 \quad t' \neq t'' \in \mathcal{A}_j \Rightarrow |t' \cap t''| < \aleph_0 = |t'|.$$

Let  $\mathcal{A}_j^+$  be a maximal almost disjoint family of countable subsets of  $\mu$  extending  $\mathcal{A}_j$ . Let  $I_j$  be the Boolean ring of subsets of  $\kappa$  generated by  $\mathcal{A}_j^+ \cup \{\{\varepsilon\} : \varepsilon < \kappa\}$  and  $B_j$  be the Boolean algebra of subsets of  $\kappa$  generated by  $I_j$ . Now

- $\odot_2$  if  $i_0, i_1 < \mu$  and  $b_0, b_1 \in [\kappa]^\theta$  and  $h$  is a one to one mapping from  $b_0$  onto  $b_1$  such that  $\alpha \in \text{Dom}(b) \Rightarrow h(\alpha) \neq \alpha$ , then for some  $t^0 \in \mathcal{A}_{i_0}^+, t^1 \in \mathcal{A}_{i_1}^+$  we have:  $t^0 \subseteq^* b_0, t^1 \subseteq^* b_1$  and  $h$  maps  $t^0$  into a co-infinite subset of  $t^1$  [why? for some  $\gamma_0 < \kappa$  the set  $b_0 \cap \bigcup_{n < \omega} a_n^{\gamma_0}$  have cardinality  $\theta$ , so without loss of generality  $b_0 \subseteq \bigcup_{n < \omega} a_n^{\gamma_0}$  and similarly for some  $\gamma_1 < \kappa$  without loss of generality  $b_1 \subseteq \bigcup_{n < \omega} a_n^{\gamma_1}$ . For  $\ell = 0, 1$  let  $b_\ell^- \in [\theta]^\theta$  be such that  $g_{\gamma_\ell}$  maps  $b_\ell^-$  onto  $b_\ell$ . Now without loss of generality  $b_0^- \cap b_1^- = \emptyset$  (recall we have to preserve "h is from  $b_0$  onto  $b_1$ ", too!). If  $b_0^- \cap b_1^- = \emptyset$  then by clause (v) of the fact some  $t_{0, \alpha}^{\gamma_0} \in \mathcal{A}_{i_0} \subseteq \mathcal{A}_{i_0}^+$  and  $t_{0, \alpha_1}^{\gamma_1} \in \mathcal{A}_{i_1} \subseteq \mathcal{A}_{i_1}^+$  will be as required in clause (α). So assume  $b_0^- = b_1^-$ , let  $b_0^* = \{\alpha \in b_0^- : h \circ g_{\gamma_0}(\alpha) \neq g_{\gamma_1}(\alpha)\}$ . If  $b_0^*$  has cardinality  $\theta$ , we get the desired conclusion (in clause (α)), so

assume  $|b_0^*| < \theta$  hence without loss of generality  $b_0^* = \emptyset$ . Also if  $\gamma_0 \neq \gamma_1$  then  $|X_{\gamma_0} \setminus X_{\gamma_1}| = \kappa$  hence we can find a non zero ordinal  $\beta \in X_{\gamma_0} \setminus X_{\gamma_1}$  and we can find an ordinal  $\alpha < 2^\theta$  such that  $(\forall \beta' < 2^\theta)[t_{2,\alpha,\beta}^\gamma \subseteq b_0^-]$  hence we can use  $t_{3,\alpha,\beta}^\gamma, t_{2,\alpha,\beta}^\gamma$ . So we have to assume  $\gamma_0 = \gamma_1$  but then  $g_{\gamma_0} = g_{\gamma_1}$  so  $h \upharpoonright (b_0 \setminus b_0^*)$  is the identity, a contradiction.]

- $\odot_3$  if  $i_0 \neq i_1$  and<sup>2</sup>  $Z \in [\kappa]^{<\theta}$  and  $h$  is a one to one function from  $\kappa \setminus Z$  onto  $\kappa \setminus Z$  then for some  $t^0 \in \mathcal{A}_{i_0}^+, t^0 \subseteq^* \text{Dom}(h)$  and  $t^1 \in \mathcal{A}_{i_1}^+$  we have:  $h''(t^0) \subseteq^* t^1, t^1 \setminus h''(t^0)$  is infinite.

[Why? Let  $Z_1 = \{\alpha \in \text{Dom}(h) : h(\alpha) \neq \alpha\}$ , so by  $\odot_2$  we know  $|Z_1| < \theta$ . We know that  $Y_{i_0} \setminus Y_{i_1}$  has cardinality  $\mu$ , hence for some  $\gamma \in Y_{i_0} \setminus Y_{i_1}$  we have  $\text{set}(\bar{a}_\gamma) \cap [Z \cup Z_1]^{\aleph_0} = \emptyset$ . So  $t_{3,\alpha,0}^\gamma \in \mathcal{A}_{i_0} \subseteq \mathcal{A}_j^+$  and  $t_{2,\alpha,0}^\gamma \in \mathcal{A}_{i_1} \subseteq \mathcal{A}_{i_1}^+$ , so  $t_{3,\alpha,0}^\gamma$  is a co-infinite subset of  $t_{2,\alpha,0}^\gamma, t_{2,\alpha,0}^\gamma \subseteq^* \kappa \setminus Z \setminus Z_0$  and  $h$  maps  $t_{3,\alpha,0}^\gamma \setminus Z \setminus Z_0$  to itself a co-infinite subset of  $t_{2,\alpha,0}^\gamma$ .]

Clearly  $\langle B_j : j < \mu \rangle$  is as required so we are done.

2) Similar proof.  $\square_{3.9}$

$\rightarrow$  MARTIN WARNS: Label ge.8 on next line is also used somewhere else (Perhaps should have used scite instead of stag?)

*3.10 Conclusion.* 1) Under the assumption of 3.9, let  $\lambda^* = \text{Ded}^+(\mu) = \text{Min}\{\lambda : \text{there is no tree with } \leq \mu \text{ nodes and } \geq \lambda \text{ branches (equivalently, a linear order of cardinality } \lambda \text{ and density } \leq \mu)\}$ . Then for any  $\lambda \in [\mu, \lambda^*)$  there is a superatomic Boolean Algebra of cardinality  $\lambda, \mu$  atoms with no automorphism moving  $\geq \theta$  atoms. 2) Assume:  $\theta$  is uncountable strong limit of cofinality  $\aleph_0$ ,  $\text{pp}_{J_{\omega}^{\text{bd}}}(\theta) = 2^\theta$  (see [Sh:g, Ch.IX,§5] why this is reasonable) and  $\kappa = (2^\theta)^{+n}, \mu = 2^\kappa$  and  $\mu < \lambda < \text{Ded}^+(\mu)$ , e.g.  $\lambda = 2^\chi$  for  $\chi = \text{Min}\{\chi : 2^\chi > \mu\}$ . Then there is a superatomic Boolean Algebra of cardinality  $\lambda$  and  $\mu$  atoms, with no automorphism moving  $\geq \theta$  atoms.

*Proof.* Combine 3.9, 3.8 and 2.1.

*Remark.* 1) So clearly in many models of ZFC we get that the bound is 1.1 cannot be improved.

2) The question is whether inductively we can get for many  $\theta$ 's the parallel of 3.9.

3) We can in **3.10** replace  $\theta$  by  $\aleph_0$  (recall for **3.10(2)**) that we can replace the use

$\rightarrow$  scite{ge.8} ambiguous  
 $\rightarrow$  scite{ge.8} ambiguous

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<sup>2</sup>by a little more care in indexing,  $Z \in [\mu]^{<\mu}$  is O.K. and we can choose  $\gamma$  such that  $\bigcup_n a_{\gamma,n} \subseteq \kappa \setminus Z \setminus Z_0$

of 3.8 by the known: there is  $\mathcal{A} \subseteq [{}^\omega 2]^{\aleph_0}$  which is MAD, every  $A \subseteq {}^\omega 2$  dense in itself contains a member of  $\mathcal{A}$ , each  $A \in \mathcal{A}$  has exactly one accumulation point.

## §4

In the bound  $\beth_4(\sigma)$ , the last exponentiation was really  $sa(\mu)$  where

**4.1 Definition.** 1)  $sa^+(\mu) = \sup\{|B|^+ : B \text{ is a superatomic Boolean Algebra with } \mu \text{ atoms}\}.$

2)  $sa(\mu) = \sup\{|B| : B \text{ is a superatomic Boolean Algebra with } \mu \text{ atoms}\}.$

3)  $sa^+(\mu, \theta) = \sup\{|B|^+ : B \text{ is a superatomic Boolean Subalgebra of } \mathcal{P}(\mu) \text{ extending } \{a \subseteq \mu : a \text{ finite or cofinite such that } a \in B \Rightarrow (a) < \theta \vee |\mu \setminus a| < \theta\}.$

4)  $sa(\mu, \theta) = \sup\{|B| : B \text{ is as in (3)}\}.$

5)  $sa^*(\theta) = \text{Min}\{\lambda : \text{cf}(\lambda) \geq \theta \text{ and if } \mu < \lambda \text{ then } sa^+(\mu, \theta) \leq \lambda\}.$

That is, by the proof of Theorem 1.1

**4.2 Claim.** *If  $B$  is a superatomic Boolean Algebra with no automorphism moving  $\geq \theta$  atoms,  $\theta = \text{cf}(\theta) > \aleph_0$  then  $|B| < sa^+(\beth_2(2^{<\theta}))$ , moreover  $|B| < sa^+(\beth_2(sa^*(\theta)))$ .*

4.3 Discussion: Now consistently  $sa(\aleph_1) < 2^{\aleph_1}$ , as [Sh 620, 8.1] show the consistency of a considerably stronger statement. It proves that e.g. if we start with  $\mathbf{V} \models \text{GCH}$  and  $\mathbb{P}$  is adding  $\aleph_{\omega_1}$  Cohen reals then in  $\mathbf{V}^{\mathbb{P}}$ , ( $2^{\aleph_0} = \aleph_{\omega_1} < 2^{\aleph_1} = \aleph_{\omega_1+1}$  and) among any  $\aleph_{\omega_1+1}$  members of  $\mathcal{P}(\omega_1)$  there are  $\aleph_{\omega_1+1}$  which form an independent family, i.e. any finite nontrivial Boolean combination of them is nonempty, in other words “ $\mathcal{P}(\omega_1)$  has  $\aleph_{\omega_1+1}$ -free precaliber in Monk’s question definition”. (Not surprising this is the same model for “no tree with  $\aleph_1$  nodes has  $2^{\aleph_1}$  branches” in [B1]). So the bound  $\beth_4(\theta)$  is not always the right ones.

**4.4 Claim.** *Assume*

- (a)  $\Upsilon = \Upsilon^{<\Upsilon} < \mu = \text{cf}(\mu) < \chi$
- (b)  $\text{cf}(\chi) = \mu$  and  $(\forall \alpha < \chi)(|\alpha|^\mu < \chi)$  and  $(\forall \alpha < \mu)(|\alpha|^{<\Upsilon} < \mu)$
- (c)  $\mathbb{Q}$  is a forcing notion of cardinality  $< \chi$  such that in  $\mathbf{V}^{\mathbb{Q}} : \mu$  is a regular cardinal  $(\forall a \in [\chi]^{<\mu})(\exists b)[a \subseteq b \in ([\chi]^{<\mu})^{\mathbf{V}}]$
- (d)  $\mathbb{P} = \{f : f \text{ a partial function from } \chi \text{ to } \{0, 1\} \text{ of cardinality } < \Upsilon\}$  order by inclusion (that is, adding a  $\chi$   $\sigma$ -Cohen).

Then in  $\mathbf{V}^{\mathbb{Q} \times \mathbb{P}}$  we have:  $(2^\sigma = 2^{<\mu} = \chi, 2^\mu = \chi^\mu = (\chi^\mu)^{\mathbf{V}}$  and)  $sa(\mu) = \chi < 2^\mu$ , moreover the Boolean Algebra  $\mathcal{P}(\mu)$  has  $\chi^+$ -free precaliber.

*Proof.* Work in  $\mathbf{V}^{\mathbb{Q}}$ , like [Sh 620, 8.1], not using “ $\mathbb{P}$  is  $\sigma$ -complete” which may fail in  $\mathbf{V}^{\mathbb{Q}}$ . □<sub>4.4</sub>

On the other hand

**4.5 Claim.** Assume  $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$  satisfies  $\lambda_{n+1} = \text{Min}\{\lambda : 2^\lambda > 2^{\lambda_n}\}$ . Then for infinitely many  $n$ 's for some  $\mu_n \in [\lambda_n, \lambda_{n+1})$  we have  $sa(\mu_n) = 2^{\mu_n} = 2^{\lambda_n}$  (in fact  $sa^+(\mu_n) = (2^{\mu_n})^+ = (2^{\lambda_n})^+$  except possibly when  $cf(2^{\lambda_n}) \leq 2^{\lambda_{n-1}}$ ).

*Proof.* By [Sh 430, 3.4] we have for infinitely many  $n$ 's  $\mu_n \in [\lambda_n, \lambda_{n+1})$  and for every regular  $\chi \leq 2^{\lambda_n} = 2^{\mu_n}$ , a tree with  $\leq \mu$  nodes,  $\lambda_n$  levels and  $\geq \chi$   $\lambda_n$ -branches.  $\square_{4.5}$

**4.6 Conclusion:** 1) Assume  $\theta$  is strong limit,  $\theta > cf(\theta) = \aleph_0$  and  $\text{Pr}(2^{2^\theta}, \theta)$  and  $\lambda < sa^+(\beth_3(\theta))$ . Then

$(*)_{\theta, \lambda}$  there is a superatomic Boolean Algebra without any automorphism moving  $\geq \theta$  atoms such that  $B$  has cardinality  $\lambda$  (and has  $\lambda$  atoms<sup>3</sup>).

2) Assume  $\text{Pr}(\beth_2, \aleph_1)$  and  $\lambda < sa^+(\beth_3)$ . Then  $(*)_{\theta, \lambda}$  holds.

*Proof.* 1) Use 3.9 and 2.1.

2) Similar only replace 3.9 by a parallel claim.

4.6

\* \* \*

→ MARTIN WARNS: Label ge.8 on next line is also used somewhere else (Perhaps should have used scite instead of stag?)

**4.7 Discussion:** [here?] Suppose  $\theta = \tau^+$  and there is a tree  $\mathcal{T}$  with  $\tau$  nodes and  $\varrho > \theta$  branches. We can build a superatomic Boolean ring  $B$  with  $|\mathcal{T}| \leq \tau$  atoms and  $\varrho$  elements, let  $Y$  be a natural set of representatives for the set of higher level atoms (i.e. not atoms). By the ultrafilter for  $y \in Y$  and say  $B \subseteq \mathcal{P}(\mathcal{T})$ ,  $\bigwedge_{t \in \mathcal{T}} [\{t\} \in$

$B]$  and  $A$  a set of  $\varrho$  elements  $\{d_y : y \in Y_0 \subseteq Y\} \subseteq [A]^{\aleph_0}$  as in §2, and use the Boolean ring  $B^\oplus$  generated by  $\{\{t\} : t \in \mathcal{T}\} \cup \{y \cup d_y : y \in Y\}$ .

We would like: every automorphism of  $B^\oplus$  moves  $\leq \tau$  atoms. If we succeed, we can continue to immitate 3.9 with the present  $\theta$ .

So it is natural to consider:

$(*)_{\tau, \varrho_1, \varrho_2, \varrho_3}$  there is a tree with  $\tau$  nodes and a set of  $\varrho_3$  branches such that an automorphism of the tree with moves  $> \tau$  branches move at least  $\varrho_2$  branches.

Restricting ourselves to  $\varrho_0$ -branches  $\varrho_0 = cf(\varrho_0) \leq \varrho_2$ ,  $2^{\varrho_0} = 2^\tau$ , we can make the superatomic Boolean Algebras quite rigid so we need

$(*)_{\tau, \varrho_1, \varrho_2}$  there is a tree  $\mathcal{T}$  with  $\tau$  nodes such that any subtree has  $\leq \varrho_1$  and  $\geq \varrho_2$  branches.

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<sup>3</sup>we can allow less atoms and less elements

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